UNIT - 03

01/08/2025 **NEGATIVE BINOMIAL DISTRIBUTION** Friday

*Negative Binomial distribution :-

For poisson distribution mean is equal to variance and for binomial distribution. Mean greater than variance. In negative binomia distribution mean less than variance. The negative binomial distribution can be derived from Binomial distribution with some modifications which can be stated below.

In where n independent Bernouli trails consider.

1. The trails are independent
2. The probability is success p is constant from trail to trail.
3. Let $p(x)$ be the negative Binomial properties that there are x failures preceeding the r^{th} success in $(x+r)$ trails. To find this last trail must be success with probability p. In the remaining $(x+r-1)$ trails we must have $(r-1)$ success with probability

$$^{x+r-1}C_{r-1}\, p^{r-1} q^{x}$$

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$$p(X=x) = {}^{x+r-1}C_{r-1}\, p^{r-1}\, q^{x} \cdot p$$

$$p(X=x) = {}^{x+r-1}C_{r-1}\, p^{r}\, q^{x}$$

Definition :- A discrete random variable X assumes the value $X=0, 1, 2 \cdots$ with probability mass function.

$$p(X=x) = \begin{cases} {}^{x+r-1}C_{r-1}\, p^{r} q^{x}, & x=0,1,2,\cdots \\ 0, & \text{otherwise} \end{cases}$$

is called Negative binomial distribution.

The parameter of negative binomial distribution are r and p. It is usually denoted by ~~follows~~ $X \sim NB(r, p)$.

* Negative Binomial forms in the calculation of moments and properties :-

Consider

$$p(X=x) = {}^{x+r-1}C_{r-1}\, p^{r}\, q^{x}$$

Consider

$${}^{x+r-1}C_{r-1}\, p^{r} q^{x} = \frac{(x+r-1)!}{(x+r-1-r+1)!\,(r-1)!} p^{r} q^{x}$$

$$= \frac{(x+r-1)!}{x!(r-1)!} p^r q^x$$

$$= \frac{(x+r-1)(x+r-2)\cdots\cdots(r+1)(r-1)!}{x!\,(r-1)!} p^r q^x$$

$$= \frac{(x+r-1)(x+r-2)\cdots\cdots(r+1)r}{x!} p^r q^x$$

$$= \frac{(-1)^x -r-r-1\cdots\cdots(-x-r+2)(-x-r+1)}{x!} p^r q^x$$

$$= (-1)^x\, {}^{-r}C_x\, p^r q^x$$

$$= (-1)^x\, {}^{-r}C_x\, p^r q^x$$

$$= {}^{-r}C_x\, p^r (-q)^x$$

$$\boxed{p(x) = {}^{-r}C_x\, p^r (-q)^x}$$

ii, if $P = \frac{1}{Q}$; $q = \frac{P}{Q}$

Note: $p(X) = \binom{-r}{x} Q^{-r} \left(\frac{-P}{Q}\right)^x$

$p = \frac{1}{Q}$, $q = \left(\frac{P}{Q}\right)$

$p+q=1$; $Q-P=1$

$$p(x) = {}^{-r}C_x \left(\frac{1}{Q}\right)^r \left(\frac{-P}{Q}\right)^x$$

$$p(x) = {}^{-r}C_x\, Q^{-r} \left(\frac{-P}{Q}\right)^x$$

Note:-

$$\boxed{\sum_{x=0}^{\infty} {}^{-r}C_x (-q)^x = (1-q)^{-r}}$$

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Moments of negative binomial distribution:-

$$\mu_1' = E(X)$$

$$= \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-q)^{x-1+1}$$

$$= \sum_{x=0}^{\infty} x \left(\frac{-r}{x}\right) \binom{-r-1}{x-1} p^r (-q)^{x-1+1}$$

$$= -rp^r \sum \binom{-r-1}{x-1} (-q)^{x-1} \cdot (-q)$$

$$= rqp^r \cdot \sum_{x=0}^{\infty} \binom{-r-1}{x-1} (-q)^{x-1}$$

$$\boxed{\mu_1' = rqp^r (1-q)^{-r-1}} \quad \left[\because \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x = (1-q)^{-r}\right]$$

$$= rqp^r p^{-r-1}$$

$$\boxed{\mu_1' = \frac{rq}{p}} = \text{Mean}$$

$$\mu_2' = E(X^2)$$

$$= E(X(X-1) + X)$$

$$= E[X(X-1)] + E(X)$$

$$= \sum_{x=0}^{\infty} x(x-1) \binom{-r}{x} p^r (-q)^x + \frac{rq}{p}$$

$$= \cancel{p^r \sum_{x=0}^{\infty} x(x-1) \left(\frac{-r}{x}\right) \frac{(-r-1)}{(x-1)} \frac{-r-2}{x-2}}$$

$$= p^r \sum_{x=0}^{\infty} x(x-1) \frac{(-r)(-r-1)}{x(x-1)} \binom{-r-2}{x-2} (-q)^x + \frac{rq}{p}$$

$$= p^r \sum_{x-2=0}^{\infty} r(r+1) \binom{-r-2}{x-2} (-q)^{x-2+2} + \frac{rq}{p}$$

$$= r(r+1)\, p^r (-q)^2 \sum_{x-2=0}^{\infty} \binom{-r-2}{x-2} (-q)^{x-2} + \frac{rq}{p}$$

$$= r(r+1)\, p^r (q)^2 (1-q)^{-r-2} + \frac{rq}{p}$$

$$= r(r+1)\, p^r q^2 p^{-r-2} + \frac{rq}{p}$$

$$\boxed{\mu_2' = \frac{r(r+1)q^2}{p^2} + \frac{rq}{p}}$$

$$Var = E(X^2) - [E(X)]^2$$

$$= \frac{r(r+1)q^2}{p^2} + \frac{rq}{p} - \left(\frac{rq}{p}\right)^2$$

$$= \frac{rq}{p}\left[\frac{(r+1)q}{p} + 1 - \frac{rq}{p}\right]$$

$$= \frac{rq}{p}\left[\frac{rq + q + p - rq}{p}\right]$$

$$= \frac{rq}{p}\left[\frac{q+p}{p}\right]$$

$$\boxed{Var = \frac{rq}{p^2}}$$

$$\mu_3' = E(X^3) = E[X(X-1)(X-2) + 3[X(X-1)] + X]$$

$$\boxed{\mu_3' = \frac{r(r+1)(r+2)q^3}{p^3} + 3\frac{r(r+1)q^2}{p^2} + \frac{rq}{p}}$$

$$\mu_4' = E(X^4) = E[X(X-1)(X-2)(X-3) + 6X(X-1)(X-2)$$

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$+7x(x-1)+x]$

$$\boxed{\mu_4' = \frac{r(r+1)(r+2)(r+3)q^4}{p^4} + 6\frac{r(r+1)(r+2)q^3}{p^3} + 7\frac{r(r+1)q^2}{p^2} + \frac{rq}{p}}$$ //

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1 + 2\mu_1'^3$$

$$= \frac{r(r+1)(r+2)q^3}{p^3} + 3\frac{r(r+1)q^2}{p^2} + \frac{rq}{p} - 3\left[\frac{r(r+1)q^2}{p^2} + \frac{rq}{p}\right]\frac{rq}{p} + 2\left(\frac{rq}{p}\right)^3$$

$$\boxed{\mu_3 = \frac{rq(1+q)}{p^3}}$$ //

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \left[\frac{r(r+1)(r+2)(r+3)q^4}{p^4} + 6\cdot\frac{r(r+1)(r+2)q^3}{p^3} + 7\frac{r(r+1)q^2}{p^2} + \frac{rq}{p}\right] - 4\left[\frac{r(r+1)(r+2)q^3}{p^3} + 3\frac{r(r+1)q^2}{p^2} + \frac{rq}{p}\right]\frac{rq}{p} + 6\left[\frac{r(r+1)q^2}{p^2} + \frac{rq}{p}\right] - 3\left(\frac{rq}{p}\right)^4$$

$$\boxed{\mu_4 = \frac{rq[p^2 + 3q(r+2)]}{p^4}}$$ //

*Skewness of Negative Binomial distribution

Coefficient of skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left[\frac{rq(1+q)}{p^3}\right]^2}{\left[\frac{rq}{p^2}\right]^3} \Rightarrow \boxed{\beta_1 = \frac{(1+q)^2}{rq}}$$ //

$$= \frac{r^2q^2(1+q)^2}{p^6} \times \frac{p^6}{r^3q^3}$$

$$\boxed{\gamma_1 = \frac{1+q}{\sqrt{rq}}}$$

* **Kurtosis of Negative Binomial distribution:**

Coefficient of Kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{rq(p^2+3q(r+2))}{p^4}}{\left(\frac{rq}{p^2}\right)^2} = \frac{\frac{rq(p^2+3q(r+2))}{p^4}}{\frac{r^2q^2}{p^4}}$$

$$\boxed{\beta_2 = \frac{p^2+3q(r+2)}{rq}} = \frac{p^2+3qr+6q}{rq}$$

$$\boxed{\gamma_2 = \frac{p^2+6q}{rq}} \quad = \frac{p^2+3qr+6q-3rq}{rq} = \frac{p^2+6q}{rq}$$

* **M.G.F of negative B.D :-**

$$M_X(t) = E\left[e^{tx}\right]$$

$$= \sum_{x=0}^{\infty} e^{tx} \binom{-r}{x} p^r (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-qe^t)^x$$

$$= p^r (1-qe^t)^{-r}$$

$$\boxed{M_X(t) = \left(\frac{p}{1-qe^t}\right)^r} \quad \text{or} \quad p^r(1-qe^t)^{-r}$$

we know that

$$P = \frac{1}{Q}, \quad q = \frac{P}{Q}$$

$$Q = \frac{1}{p}, \quad P = qQ = q \cdot \frac{1}{p} = \frac{q}{p}$$

$$M_x(t) = \left(\frac{p}{1-qe^t}\right)^r = \left[\frac{1}{\frac{1-qe^t}{p}}\right]^r$$

$$= \left[\frac{1}{\left(\frac{1}{p} - \frac{q}{p}e^t\right)}\right]^r$$

$$= \left(\frac{1}{Q - Pe^t}\right)^r$$

$$\boxed{M_x(t) = (Q - Pe^t)^{-r}}$$

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V. Imp

Mean and Variance through M.G.F:-

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0} \qquad \mu_r' = \left(\frac{d^r}{dt^r} M_x(t)\right)_{t=0}$$

$$= \left[\frac{d}{dt} p^r (1-qe^t)^{-r}\right]_{t=0}$$

$$= \left[p^r(-r)(1-qe^t)^{-r-1} \cdot (-qe^t)\right]_{t=0}$$

$$= p^r \cdot rq\,(1-q)^{-r-1}$$

$$= p^r \cdot rq\, p^{-r-1}$$

$$\boxed{\mu_1' = \frac{rq}{p} = \text{Mean}}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} \cdot M_X(t)\right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2}\, p^r (1-qe^t)^{-r}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{d}{dt}\, p^r (1-qe^t)^{-r}\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\, p^r (-r)(1-qe^t)^{-r-1}(-qe^t)\right]_{t=0}$$

$$= p^r rq \left[\frac{d}{dt}(1-qe^t)^{-r-1}(-qe^t)\right]_{t=0}$$

$$= p^r rq\left[(1-qe^t)^{-r-1} \cdot e^t + e^t(-r-1)(1-qe^t)^{-r-2}(-qe^t)\right]_{t=0}$$

$$= p^r rq\left[p^{-r-1} + (r+1)p^{-r-2}q\right]$$

$$= rqp^r p^{-r-1} + r(r+1)q^2 p^r \cdot p^{-r-2}$$

$$\boxed{\mu_2' = \frac{rq}{p} + r(r+1)\frac{q^2}{p^2}}$$

$$\text{Var } \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{rq}{p} + r(r+1)\frac{q^2}{p^2} - \left(\frac{rq}{p}\right)^2$$

$$= \frac{rq}{p}\left[1 + \frac{(r+1)q}{p} - \frac{rq}{p}\right]$$

$$= \frac{rq}{p}\left[\frac{p + rq + q - rq}{p}\right]$$

$$= \frac{rq}{p}\left[\frac{p+q}{p}\right]$$

$$\boxed{\text{Var } \mu_2 = \frac{rq}{p^2}}$$

*Cummulant generating function of NBD:-

we know that

$$K_X(t) = \log(M_X(t))$$

$$= \log\left[p^r(1-qe^t)^{-r}\right]$$

$$= r\log p - r\log\left[1 - q\left(1 + t + \frac{t^2}{2!} + \dots\right)\right]$$

$$= \left[r\log + r\log\left\{q\left(1+t+\frac{t^2}{2!} + \dots\right) + \frac{\left[q\left(1+t+\frac{t^2}{2!}+\dots\right)\right]^2}{2} + \dots\right\}\right.$$

Cummulants

$$\text{Mean} = K_1 = \mu_1' = \frac{rq}{p}$$

$$Var = K_2 = \mu_2 = \frac{rq}{p^2}$$

$$K_3 = \mu_3 = \frac{rq(1+q)}{p^3}$$

$$K_4 = \frac{rq}{p^2}\left(1 + \frac{6q}{p^2}\right)$$

$$\mu_4 = K_4 + 3K_2^2$$

$$= \frac{rq}{p}\left(\frac{1+6q}{p^2}\right) + 3\left(\frac{rq}{p}\right)^2$$

$$\boxed{\mu_4 = \frac{rq\,[p^2 + 3q(r+2)]}{p^4}}$$

Prop * <u>characteristic function of NBD</u> :-

we know that

$$\phi_x(t) = E(e^{itx})$$

$$= \sum_{x=0}^{\infty} e^{itx} \cdot p(x)$$

$$= \sum_{x=0}^{\infty} e^{itx} \binom{-r}{x} p^r (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-qe^{it})^x$$

$$= p^r (1-qe^{it})^{-r}$$

$$\boxed{\phi_x(t) = \left(\frac{p}{1-qe^{it}}\right)^r}$$

* Additive property of NBD:-

Statement:

Sum of 'n' independent negative binomial variates is also a negative binomial variate.

proof:

Let $X_1, X_2, \dots X_n$ be n independent negative binomial variates with the parameter $(r_1, p), (r_2, p) \dots (r_n, p)$ respectively.

M.G.F's are $M_{X_1}(t) = p^{r_1}(1-qe^t)^{-r_1}$

$M_{X_2}(t) = p^{r_2}(1-qe^t)^{-r_2}$

$\vdots$

$M_{X_n}(t) = p^{r_n}(1-qe^t)^{-r_n}$

Consider, the M.G.F of $\sum_{i=1}^{n} x_i$:-

$$M_{\sum_{i=1}^{n} x_i}(t) = M_{x_1+x_2+\dots x_n}(t)$$

$$= M_{x_1}(t) \cdot M_{x_2}(t) \dots M_{x_n}(t)$$

$$= p^{r_1}(1-qe^t)^{-r_1} \cdot p^{r_2}(1-qe^t)^{-r_2} \dots p^{r_n}(1-qe^t)^{-r_n}$$

$$= p^{r_1+r_2+\dots+r_n} \boxed{(1-qe^t)^{-(r_1+r_2+\dots+r_n)}}$$

This is looking to be M.G.F of negative Binomial distribution.

Hence, by uniqueness theorem of M.G.F of $\sum_{i=1}^{n} x_i$ follows NBD with the parameters. $(r_1 + r_2 + \cdots + r_n, p)$

Negative Binomial distribution satisfies the additive property.

* <u>Recurrence relation for probabilities of Negative Binomial distribution</u>:-

$$p(x) = \binom{x+r-1}{r-1} p^r q^x$$

$$p(x+1) = \binom{x+r}{r-1} p^r q^{x+1}$$

$$\frac{p(x+1)}{p(x)} = \frac{\binom{x+r}{r-1} p^r q^{x+1}}{\binom{x+r-1}{r-1} p^r q^x}$$

$$= \frac{(x+r)!}{(r-1)!\,(x+1)!} \cdot \frac{(r-1)!\,x!}{(x+r-1)!} \cdot q.$$

$$= \frac{x+r}{x+1} \cdot q$$

$$p(x+1) = \frac{x+r}{x+1} q\, p(x)$$

* limiting case of Negative Binomial distribution to Normal distribution:-

The M.G.F of standard Negative Binomial distribution is $M_z(t) = \left[Qe^{\frac{tP}{\sqrt{nPQ}}} - Pe^{\frac{tQ}{\sqrt{nPQ}}}\right]^{-n}$

$$= \left\{Q\left[1+\frac{tP}{\sqrt{nPQ}} + \frac{t^2P^2}{2!\,nPQ} + \frac{t^3P^3}{3!(nPQ)^{3/2}} + \cdots\right]\right.$$

$$\left. -P\left[1+\frac{tQ}{\sqrt{nPQ}} + \frac{t^2Q^2}{2!\,nPQ} + \frac{t^3Q^3}{3!\,(nPQ)^{3/2}} + \cdots\right]\right\}^{-n}$$

$$= \left\{(Q-P) + \frac{t^2}{2n}(P-Q) + O(n^{-3/2})\right\}^{-n}$$

$$M_Z(t) = \left\{1 - \frac{t^2}{2n} + O(n^{-3/2})\right\}^{-n} \quad (\because Q-P=1)$$

where $O(n^{-3/2})$ is the term containing the power 3/2 and more of n in denominator.

Now consider logarithm

$$\log M_Z(t) = -n\log\left[1-\frac{t^2}{2n} + O(n^{-3/2})\right]$$

$$= n\log\left[1-\frac{t^2}{2n} - O(n^{-3/2})\right]$$

$$= n\left[\left(\frac{t^2}{2n} - O(n^{-3/2})\right) + \frac{1}{2}\left[\frac{t^2}{2n} - O(n^{-3/2})\right]^2 + \cdots\right]$$

$$\log M_Z(t) = \frac{t^2}{2} - o(n^{-1/2})$$

apply limit as $n \to \infty$

$$\lim_{n\to\infty} \log M_Z(t) = \frac{t^2}{2}$$

Consider exponential on both sides

$$\lim_{n\to\infty} M_Z(t) = e^{t^2/2}$$

This is M.G.F of standard normal variate.

Hence, By uniqueness theorem of M.G.F standard Negative Binomial variates approaches standard normal distribution, i.e., Z is Asymptotically normal as $n \to \infty$. tends to ∞

Hence we can conclude that negative binomial distribution tends to normal distribution for large values of n i.e., as $n \to \infty$.

S.M. Shams ~~//~~